GAUSSIAN PROCESSES
EXERCISE SHEET 9: GAUSSIAN FREE FIELD

Exercise 1.

(1) Since T is invertible and linear, the change-of-variables formula gives

$$f_{Y,Z}(y,z) = f_X(T^{-1}(y,z))\,J,$$

where $J = |\det(DT^{-1})|$ is the constant Jacobian determinant of the linear map.

The conditional density of Z given $Y = b$ is defined by

$$f_{Z|Y=b}(z) = \frac{f_{Y,Z}(b,z)}{\displaystyle\int_{\mathbb{R}^{n-k}} f_{Y,Z}(b,z')\,dz'}.$$

Substituting the expression for $f_{Y,Z}$ and noting that J is constant, we obtain

$$f_{Z|Y=b}(z) = \frac{f_X(T^{-1}(b,z))}{\displaystyle\int_{\mathbb{R}^{n-k}} f_X(T^{-1}(b,z'))\,dz'}.$$

(2) We let $X : E \to \mathbb{R}$ be a standard Gaussian defined on the edges of the graph. For the transformation $(Y,Z) = T(X)$, we define $Z : V \setminus V_\partial \to \mathbb{R}$ so that $Z(v)$ is the sum of X along any path from a boundary vertex to v; this corresponds to the height function h as defined in Part A of the lecture. We set $Y = 0$ to encode the linearly independent constraints described in Part B. One can verify that T is a valid invertible transformation.

Noting that $T^{-1}(0,h)(e) = h(v) - h(w)$ for an edge e between vertices v and w, we then obtain the formula in Part A directly from the density of the standard Gaussian.

□

Exercise 2.

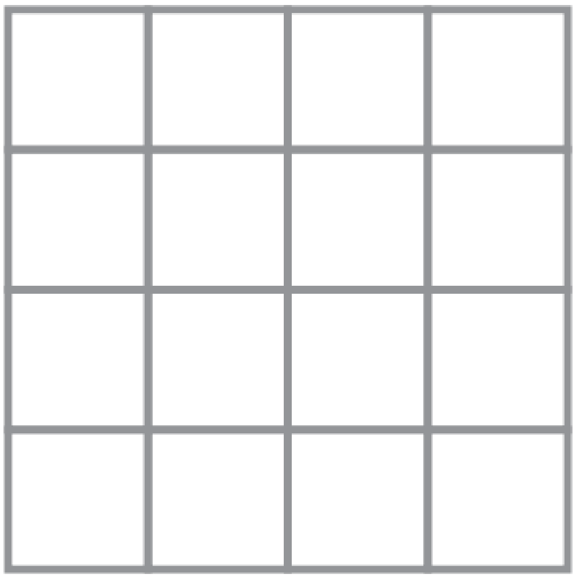
One simply notes that

$$\mathbb{E}\left[\sum_{t=1}^{\tau} \mathbf{1}_{\{W_v(t)=w\}}\right] = \frac{1}{d_v}\sum_{v'\sim v} \mathbb{E}\left[\sum_{t=0}^{\tau} \mathbf{1}_{\{W_{v'}(t)=w\}}\right],$$

by conditioning on the first step of the walk. Indeed, if we take $v' = W_v(1)$, then the trajectory $\{W_v(1),\dots,W_v(\tau)\}$ has the same law as a new random walk started from v', which yields the identity.

Remark: Using this exercise, we can directly show the equivalence between A and C in the lecture last week. □

Exercise 3.



Note that the law of g_2 is precisely the conditional law of g restricted to the orthogonal complement of the subspace described in Part B of last weeks lecture. One can verify that, within this orthogonal complement, the following constraints hold for every $v \in V \setminus V_\partial$:

$$\sum_{e \text{ starts at } v} g(e) = 0.$$

Moreover, these conditions already characterize the entire orthogonal complement.

Viewed in the dual space, these constraints correspond exactly to the defining relations of the gradient field of the Gaussian Free Field on faces discussed in Part B of the lecture.

Remark: For more reference, one can check the so-called Hodge decomposition. In this exercise, g_1 is the curl-free component and g_2 is the divergence-free component. There's no harmonic component in this setting. □